

Superquasicrystals: self-similar-ordered structures with non-crystallographic point symmetries

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2005 J. Phys. A: Math. Gen. 38 L199

(<http://iopscience.iop.org/0305-4470/38/13/L01>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.66

The article was downloaded on 02/06/2010 at 20:06

Please note that [terms and conditions apply](#).

LETTER TO THE EDITOR

Superquasicrystals: self-similar-ordered structures with non-crystallographic point symmetries

Komajiro Niizeki¹ and Nobuhisa Fujita²¹ Department of Physics, Graduate School of Science, Tohoku University, Sendai 980-8578, Japan² Structural Chemistry, Arrhenius Laboratory, Stockholm University, 10691 Stockholm, Sweden

E-mail: niizeki@cmpt.phys.tohoku.ac.jp

Received 4 November 2004

Published 14 March 2005

Online at stacks.iop.org/JPhysA/38/L199**Abstract**

We present a systematic method of constructing limit-quasiperiodic structures with non-crystallographic point symmetries. Such structures are different aperiodic-ordered structures from quasicrystals, and we call them ‘superquasicrystals’. They are sections of higher dimensional limit-periodic structures constructed on ‘super-Bravais-lattices’. We enumerate important super-Bravais-lattices. Superquasicrystals with strong self-similarities form an important subclass. The simplest example is a two-dimensional octagonal superquasicrystal.

PACS numbers: 61.44.Br, 61.50.Ah, 02.20.Hj

Quasicrystals (QCs) as well as the Penrose patterns are aperiodic-ordered structures having not only long-range positional orders but also non-crystallographic point symmetries (NCPSs) [1, 2]. Moreover, they have self-similarities, and are produced by substitution rules (equivalently, inflation rules). They can be described alternatively as quasiperiodic point sets, given as sections of higher dimensional crystals, or *hypercrystals* [3–5]. From the viewpoint of aperiodic-ordered structures, however, there can be another type of self-similar structures with long-range positional orders, namely, *limit-quasiperiodic structures* (LQPSs), which are sections of higher dimensional limit-periodic structures [6]. An LQPS is usually given as the set of vertices of an aperiodic tiling generated by a substitution rule with the *Pisot property* [6–8]. On the other hand, a limit periodic structure and LQPSs are incorporated in *model sets*, which are projections of some transcendental host structures (see [9] and references cited therein). Since a limit-quasiperiodic structure is not periodic, it as well as a QC allows any NCPS, and we shall call such a structure a ‘superquasicrystal’ (SQC). An SQC is a section of a higher dimensional limit-periodic structure, which we shall call a *supercrystal*; we must distinguish the supercrystal from a hypercrystal. Regrettably, no concrete example

of SQCs has been known previously. In this letter, the following four points will be discussed. (i) A *super-Bravais-lattice*, which is a geometric object in a Euclidean space, is introduced as a host structure on which supercrystals are constructed. (ii) Important super-Bravais-lattices with NCPs are enumerated. (iii) It is shown that an SQC can have a self-similarity characterized by a Pisot number which is severely restricted by the super-Bravais-lattice. (iv) As the simplest example, an octagonal SQC is presented.

We begin with a brief review of QCs (for a fuller review, see [10]). The point groups which are important in application to physical QCs include D_8 (octagonal), D_{10} (decagonal) and D_{12} (dodecagonal) in two dimensions (2D) and Y_h (icosahedral) in 3D; all these point groups are isometric. A QC having one of these point groups is a d -dimensional section of a hypercrystal in the $2d$ -dimensional space E_{2d} with $d = 2$ or 3 being the number of dimensions of the physical space E_d . The entities of the hypercrystal are not atoms but geometric objects called *hyperatoms* (atomic surfaces, windows, etc), which are bounded open domains in the internal space, E_d^\perp , i.e., the orthogonal complement to E_d . We consider a hypercrystal constructed by locating one kind of hyperatoms onto the sites of a Bravais lattice $\hat{\Lambda}$ in E_{2d} . There exists only one Bravais lattice for each of the three 2D point groups, but there exist three types of icosahedral Bravais lattices, namely, primitive, face-centred and body-centred types. An infinite number of different QCs are obtained from a single hypercrystal by choosing different sections which are parallel to E_d , and they form a so-called *locally isomorphic class* (LI class) [3]. The point group \hat{G} of $\hat{\Lambda}$ is isomorphic with a non-crystallographic point group G in d dimensions. More precisely, $\hat{G} = G \oplus G^\perp$, where G and G^\perp ($\simeq G$) are groups acting onto E_d and E_d^\perp , respectively. Let π and π^\perp be the projectors projecting E_{2d} onto E_d and E_d^\perp , respectively. Then, $\Lambda := \pi \hat{\Lambda}$ and $\Lambda^\perp := \pi^\perp \hat{\Lambda}$ are modules with d generators and are dense sets. They have a scaling symmetry whose scale is a quadratic irrational, τ , where $\tau = 1 + \sqrt{2}$, $\frac{1}{2}(1 + \sqrt{5})$ or $2 + \sqrt{3}$ for the case of the three 2D point groups, D_8 , D_{10} or D_{12} , respectively, while $\tau = 2 + \sqrt{5}$ for the primitive-icosahedral case but $\tau = (1 + \sqrt{5})/2$ for the other two icosahedral cases; τ is a Pisot number and also a unit in $\mathbf{Z}[\tau] := \{n + m\tau \mid n, m \in \mathbf{Z}\}$. Let φ be the scaling operation with the ratio τ . Then, any member of the infinite group, $\mathcal{G} := \langle G, \varphi \rangle$, generated by G and φ is an automorphism of Λ . A QC is ‘point diffractive’ in the sense that its structure factor is composed only of Bragg spots. The position vectors of the Bragg spots form the Fourier module, which corresponds to the reciprocal lattice for a periodic crystal. The Fourier module is given by $\Lambda^* := \pi \hat{\Lambda}^* \subset E_d^*$ with $\hat{\Lambda}^*$ being the reciprocal lattice of $\hat{\Lambda}$ and E_d^* the dual space to E_d .

We have already seen several examples of triads of objects, $\{X, X^\perp, \hat{X}\}$, associated with the three worlds, E_d , E_d^\perp and E_{2d} , where $X = \pi \hat{X}$ and $X^\perp := \pi^\perp \hat{X}$. Any member of a triad uniquely determines the remaining two; in particular, $\hat{X} = \pi^{-1}X$ is the lifted version of X . Many relationships are isomorphic among the three worlds, and a relationship in one of the three worlds can be readily translated to those in the other two [4]; if X and X^\perp above are sets, the symbol ‘ \perp ’ turns out a bijection from X onto X^\perp . There exists an important triad of linear maps $\{\varphi, \varphi^\perp, \hat{\varphi}\}$ with φ^\perp being a scaling with the ratio $\bar{\tau}$, the algebraic conjugate of τ , whereas $\hat{\varphi} = \varphi \oplus \varphi^\perp$ is an automorphism of $\hat{\Lambda}$: $\hat{\varphi} \hat{\Lambda} = \hat{\Lambda}$. It follows that a QC has a self-similarity with ratio τ [4, 11].

Another important triad of linear maps is associated with a bi-similarity transformation $\hat{\sigma} = \sigma \oplus \sigma^\perp$, where σ and σ^\perp are similarity transformations acting onto E_d and E_d^\perp , respectively, while $\hat{\sigma}$ satisfies $\hat{\Lambda}_1 := \hat{\sigma} \hat{\Lambda} \subset \hat{\Lambda}$. $\hat{\Lambda}_1$ is the last member of the triads $\{\Lambda_1, \Lambda_1^\perp, \hat{\Lambda}_1\}$, in which $\Lambda_1 = \sigma \Lambda$ and $\Lambda_1^\perp = \sigma^\perp \Lambda^\perp$ are *similar submodules* (SSMs) of Λ and Λ^\perp , respectively. Moreover, Λ_1 as well as Λ is invariant against the action of \mathcal{G} . We shall denote by m the index of Λ_1 in Λ : $m := [\Lambda : \Lambda_1]$. The scaling transformation φ is a special similarity transformation since it is invertible and, consequently, $m = 1$. However,

if $m > 1$, Λ is a larger set than Λ_1 , and hence σ is not invertible in Λ . $\hat{\Lambda}_1$ in this case is a sublattice of $\hat{\Lambda}$ or, in the terminology of crystallography, a superlattice of $\hat{\Lambda}$, and is divided into m equivalent sublattices to $\hat{\Lambda}_1$. We shall call $\hat{\Lambda}_1$ a quasi-similar superlattice of $\hat{\Lambda}$. The similarity transformation σ combining an SSM Λ_1 with Λ by $\Lambda_1 = \sigma\Lambda$ is, however, not uniquely determined by Λ_1 because any member of the set $\mathcal{G}(\Lambda_1) := \sigma\mathcal{G} = \mathcal{G}\sigma$ satisfies the same condition. The set of all the similarity ratios of the members of $\mathcal{G}(\Lambda_1)$ is given by $\{|\sigma|\tau^k \mid k \in \mathbf{Z}\}$ with $|\sigma|$ being the similarity ratio of σ . Let $\hat{\mathcal{B}}$ be the set of all the quasi-similar superlattices of $\hat{\Lambda}$. Then, it is a member of a triad, $\{\mathcal{B}, \mathcal{B}^\perp, \hat{\mathcal{B}}\}$, and \mathcal{B} is the denumerable set of all the SSMs of Λ . The set, Σ , of all the similarity transformations associated with \mathcal{B} form a semigroup. We may expect that there exists a bijection between \mathcal{B} and the quotient semigroup, Σ/\mathcal{G} . This is true provided that, for the case of the 2D point group D_{12} , \mathcal{G} is slightly modified, as done in the next paragraph.

Prior to proceeding to the subject of super-Bravais-lattices, we shall investigate in more detail SSMs of Λ for the case of the 2D point group D_p with $p = 8, 10$ or 12 . An important member of \mathcal{B} is the one written as $\sigma_p = |\sigma_p|\rho_{2p}$ with $|\sigma_p| := 2\cos(\pi/p)$ and ρ_k being the rotation operation through $2\pi/k$ [11, 12]. The index m_p of the SSM, $\sigma_p\Lambda$, is equal to 2, 5 or 1 for $p = 8, 10$ or 12 , respectively. Since σ_{12} is invertible and $|\sigma_{12}| = \sqrt{\tau}$, we have to redefine the map φ for $p = 12$ by σ_{12} and, correspondingly, the automorphism group \mathcal{G} of Λ is redefined. There exist two types of SSMs: we call $\Lambda_1 = \sigma\Lambda$ a type I or II SSM if σ is chosen to be a simple scaling or not, respectively. A complete discussion for possible SSMs has been made in [12]. A type I SSM is written with a positive number $v \in \mathbf{Z}[\tau]$ as $v\Lambda$, and its index is given by $m = [N(v)]^2$ with $N(v) := v\bar{v}$. A simplest SSM for $p = 12$, for example, is given by $(1 + \sqrt{3})\Lambda$, whose index is equal to 4. On the other hand, type II SSMs are somewhat complicated. We shall call a type II SSM proper if the point group D_p of Λ leaves it invariant. If it is not proper, the common point group between it and Λ is equal to C_p . In this letter, we shall ignore ‘improper’ type II SSMs (cf [11]). Then, there exist no type II SSMs for $p = 12$. A simplest type II SSM for $p = 8$ or 10 is $\sigma_p\Lambda$. A general type II SSM is written with $v \in \mathbf{Z}[\tau]$ as $v\sigma_p\Lambda$ and $m = m_p[N(v)]^2$. We may write $\sigma := v\sigma_p = |\sigma|\rho_{2p}$ with $|\sigma| = v|\sigma_p|$. Then, $\sigma^2\Lambda$ is identical to the type I SSM, $|\sigma|^2\Lambda$, because $(\rho_{2p})^2 (= \rho_p) \in D_p$. It follows that $|\sigma|^2 \in \mathbf{Z}[\tau]$. In particular, $|\sigma_8|^2 = 2 + \sqrt{2} = \sqrt{2}\tau$, and $\sigma_8^2\Lambda = \sqrt{2}\Lambda$ for $p = 8$ because $\tau\Lambda = \Lambda$. Similarly, $|\sigma_{10}|^2 = \sqrt{5}\tau$ and $\sigma_{10}^2\Lambda = \sqrt{5}\Lambda$ for $p = 10$. Note that $|\sigma_p| \notin \mathbf{Z}[\tau]$.

The modules $\Lambda_n := \sigma^n\Lambda, \forall n \in \mathbf{N}$, satisfy $\Lambda \supset \Lambda_1 \supset \Lambda_2 \supset \dots$ and $[\Lambda : \Lambda_n] = m^n$. We shall denote by ‘ $\stackrel{n}{\equiv}$ ’ the equivalence relationship introduced into Λ by the residue module, Λ/Λ_n . Its important property is the following: if $\ell \stackrel{n}{\equiv} \ell'$ for $\ell, \ell' \in \Lambda$, $\ell \stackrel{n'}{\equiv} \ell', \forall n' \leq n$. As a consequence, Λ becomes a normed module if a non-Archimedean norm of $\ell \in \Lambda$ is defined by $\|\ell\| := 1/2^n$ with n being the largest number satisfying $\ell \stackrel{n}{\equiv} 0$. It is just a metric space called an *inverse system*; different vectors in Λ are ‘coloured’ by different colours, and Λ is regarded as a ‘coloured module’ with an infinite number of colours. This structure can be transferred to $\hat{\Lambda}$, yielding a $2d$ -dimensional coloured lattice, \mathcal{L} . We shall call \mathcal{L} a *super-Bravais-lattice* because different supercrystals are constructed on it as shown shortly. Since $\hat{\Lambda}_n := \pi^{-1}\Lambda_n$ is superlattice of $\hat{\Lambda}$, \mathcal{L} is, intuitively, a recursive superlattice structure. The map σ as well as the group \mathcal{G} acts naturally onto \mathcal{L} , and $\sigma\mathcal{L} \subset \mathcal{L}$. We shall turn our arguments to the dual space (or the reciprocal space). The modules $\Lambda_n^* := \sigma^{-n}\Lambda^*, \forall n \in \mathbf{Z}$, satisfy $\Lambda_n^* = \sigma\Lambda_{n+1}^* \subset \Lambda_{n+1}^*$. The denumerable set $\hat{\Lambda}_\infty^* := \pi^{-1}\Lambda_\infty^*$ with $\Lambda_\infty^* := \Lambda_0^* \cup \Lambda_1^* \cup \Lambda_2^* \cup \dots$ is the dual module to \mathcal{L} . It is not finitely generated in contrast to Λ^* . It is invariant against the action of the infinite group, $\langle G, \sigma \rangle$. We can divide Λ_∞^* into the disjoint sets, $\Delta_n := \Lambda_n^* - \Lambda_{n-1}^*, n \in \mathbf{Z}$, which are not modules. They are invariant against the action of G and satisfy $\Delta_n = \sigma\Delta_{n+1}$ and

$\Delta_n \subset \Lambda_n^*$. We may write as $\Lambda_\infty^* = \Lambda_0^* + \Delta_1 + \Delta_2 + \dots$ because Λ_0^* is disjoint with Δ_n , $n > 0$. Any vector in Δ_n is indexed with the generators of Λ_n^* by $2d$ integers.

In order to construct a supercrystal on the super-Bravais-lattice, \mathcal{L} , we need a set \mathcal{A} consisting of all the allowed hyperatoms, whose diameters are assumed to be bounded by a positive number. We can identify a hyperatom with its characteristic function on E_d^\perp , and \mathcal{A} is embedded into L^1 , the function space with the $p = 1$ norm. Let α be a map from Λ into \mathcal{A} . Then, a supercrystal is specified by \mathcal{L} and α ; we shall denote it by $\mathcal{S}(\mathcal{L}, \alpha)$. The hyperatoms in a supercrystal are *not uniform* but their shapes, sizes and/or orientations are determined by the colours of the relevant sites. We assume α to be *uniformly continuous* in the sense that, for any $\varepsilon > 0$, there exists an integer n such that $\|\alpha_\ell - \alpha_{\ell'}\|_1 < \varepsilon$, $\forall \ell, \ell' \in \Lambda$ with $\ell \stackrel{n}{\equiv} \ell'$, where $\alpha_\ell \in \mathcal{A}$ stands for the image of $\ell \in \Lambda$. The point group of the supercrystal is identical to \hat{G} if $\alpha G = G^\perp \alpha$ and \mathcal{A} is invariant against the action of G^\perp . The map fulfils these conditions if α_ℓ is, for instance, a regular p -gon whose size is given by $f(\|\ell\|)$ with $f(x)$ being a continuous function bounded from both sides by two positive numbers. Let α be a special map satisfying $\alpha_\ell = \alpha_{\ell'}, \forall \ell, \ell' \in \Lambda$ with $\ell \stackrel{n}{\equiv} \ell'$ for a specified positive integer n . Then, $\mathcal{S}(\mathcal{L}, \alpha)$ degenerates into a hypercrystal whose translational group is given by $\hat{\Lambda}_n$. If the *supremum norm* is introduced into the ‘function space’ of maps, $\{\alpha\}$, a supercrystal can be approximated in any precision in this norm by a hypercrystal to be called an *approximant hypercrystal*. This means that the supercrystal is limit periodic [6].

An SQC obtained from the supercrystal, $\mathcal{S}(\mathcal{L}, \alpha)$, is parametrized by the phase vector, $\phi \in E_d^\perp$, specifying the ‘level’ at which the section of the supercrystal is taken. It is represented as $S(\mathcal{L}, \alpha, \phi) := \mathcal{S}(\mathcal{L}, \alpha) \cap (\phi + E_d)$, which is a discrete subset of Λ . If identical point scatters with the unit scattering strength are located on the sites of the SQC, we obtain a scatterer field on E_d :

$$s(\mathbf{x}) = \sum_{\ell \in \Lambda} \alpha_\ell(\phi - \ell^\perp) \delta(\mathbf{x} - \ell) \quad (1)$$

with $\alpha_\ell(\mathbf{x}^\perp)$ being the characteristic function of α_ℓ . This form of the scatterer field can be generalized for the case where α is a generic uniformly continuous map from Λ into L^1 ; the scattering strength of a site may now depend on the ‘colour’ of the site. This generalized scatterer field is point diffractive because it can be approximated in any precision in the supremum norm by one of *its quasiperiodic approximant*. Thus, an SQC is a perfectly ordered structure with a long-range order. An SQC is, in fact, a model set. This is shown by identifying α with the set $\{(\alpha_\ell, \ell) \mid \ell \in \Lambda\} (\subset E_d^\perp \times \Lambda)$, which is basically a *window* in the theory of the model set [9].

The Fourier module of the SQC is given by Λ_∞^* , and each Bragg spot is indexed by $(2d + 1)$ integers; the last index specifies the superlattice order n . We should emphasize that the Fourier module of $S(\mathcal{L}, \alpha, \phi)$ is determined solely by \mathcal{L} , the super-Bravais-lattice. The Fourier transform of the distribution, equation (1), is a distribution in E_d^* :

$$s^*(\mathbf{Q}) = \sum_{\ell^* \in \Lambda_\infty^*} \alpha_{\ell^*}^*(-(\ell^*)^\perp) \delta(\mathbf{Q} - \ell^*), \quad (2)$$

where α^* stands for a map from Λ_∞^* into L^1 defined on $(E_d^*)^\perp$. Since $\mathcal{A} \subset L^1$, we can define an average over a set of hyperatoms. Then, for $n \in \mathbf{N}$, we can define naturally the n th order averaged hypercrystal, in which the hyperatom assigned to $\hat{\ell} \in \hat{\Lambda}_n$ is the average over all the hyperatoms on the residue class $\hat{\ell} + \hat{\Lambda}_n$. It can be readily shown that the Fourier transform of the scatterer field of the averaged QC is identical to the one obtained from equation (2) by restricting the summand to Λ_n^* . This allows us to determine the map α^* .

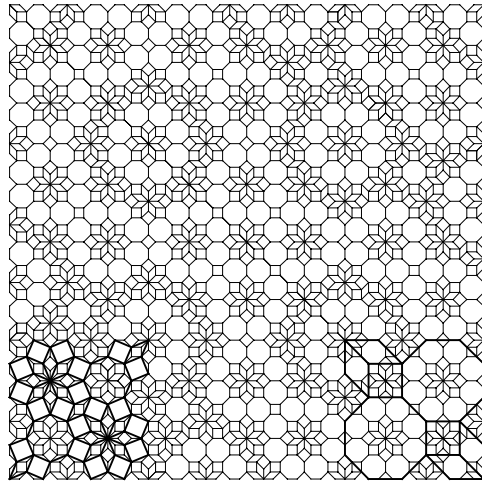


Figure 1. An octagonal SQC, S , associated with a tiling produced by a substitution rule for the three kinds of tiles. The bottom left corner is the centre of the exact octagonal symmetry. This centre is the origin of E_2 . A part of its inflation, $\sigma_8 S$, is shown with thick lines at the bottom left, while a part of the double inflation, $(\sigma_8)^2 S = (2 + \sqrt{2})S$, is at the bottom right.

An SQC has always a self-similarity in the sense that it includes a subset which is geometrically similar to itself. Its proof is basically similar to the one which was made in [14] for the case of uniform hyperatoms. The key concept in the proof is *Pisot maps*. A similarity transformation σ is called a Pisot map if $|\sigma^\perp| < 1$; this implies $|\sigma| > 1$ because $m = (|\sigma||\sigma^\perp|)^d \geq 1$. The map, φ , is a Pisot map because $|\varphi^\perp| = \tau^{-1} < 1$. Since $\varphi\mathcal{G}(\Lambda_1) = \mathcal{G}(\Lambda_1)$, $\mathcal{G}(\Lambda_1)$ includes an infinite number of Pisot maps. A similar subset of $S := S(\mathcal{L}, \alpha, 0)$ is written as $S_1 := \sigma S$ with $\sigma \in \mathcal{G}(\Lambda_1)$ being a Pisot map [14]. We may say that the SQC, S , is strongly self-similar if it and its inflation, S_1 , are mutually locally derivable (MLD: for MLD see [13]). This is not necessarily the case for a generic α . Regrettably, we have yet found no condition for α such that a strongly self-similar SQC is obtained. A weak self-similarity is of no physical interest as discussed in [14].

Since $S_1 \subset \Lambda_1$, the inflated SQC come only from one of m submodules into which Λ is divided. More generally, the n th inflation, $S_n := \sigma^n S$, is a subset of Λ_n . This is in sharp contrast to the case of a QC, for which the inflated QC come evenly from different submodules on account of $\varphi\Lambda = \Lambda$. The self-similarity ratio $|\sigma|$ (or its square $|\sigma|^2$) is a Pisot number in $\mathbf{Z}[\tau]$ if the map σ is of the type I (or II). However, it is not a unit in contrast to the self-similarity ratio of a QC. In particular, the smallest value of $|\sigma|$ is given by $|\sigma_8|$, $|\sigma_{10}|\tau$ or $1 + \sqrt{3}$ for the case of the 2D point group D_p with $p = 8, 10$ or 12 , respectively. It should be emphasized that self-similarity of an SQC as well as a QC is a natural consequence of its NCPS.

A limit-quasiperiodic tiling produced by a substitution rule is necessarily strongly self-similar. Properties of 2D and 3D tilings of this type have been extensively investigated in [6], and a recursion formula determining the Fourier transform of the relevant scatterer field is presented. A limit-quasiperiodic tiling is considered to be an SQC only if it has an NCPS. However, such a tiling has not been known so far³.

We have discovered a few examples of substitution rules which generate SQCs. The simplest of them is the octagonal SQC in figure 1 (cf [15]). The relevant SSM is given by $\sigma_8\Lambda$.

³ A tiling in [15] is approximately an SQC but its ‘octagonal symmetry’ is broken as pointed out in [14].

We have not yet determined the exact form of the relevant map, α , of this SQC. Our preliminary investigation strongly indicates, however, that the hyperatoms are topologically discs with fractal boundaries. Strong Bragg spots of the SQC are located on Λ^* , the Fourier module of an octagonal QC, while those on Δ_n , $n > 0$ are all weak; the latter are superlattice reflections in the terminology of crystallography. Interestingly, we have noted that the distribution of Bragg spots exhibits a pattern strongly reflecting the symmetry of σ_8 .

The section, S_s , of the octagonal SQC in figure 1 through the horizontal line at the bottom yields a 1D limit-quasiperiodic tiling with two intervals, S and L , whose lengths satisfy $|L|/|S| = \sqrt{2}$. The tiling is produced by the substitution rule: $S \rightarrow SLS$, $L \rightarrow LSSL$. It is, alternatively, given as a section of a 2D limit-periodic structure which is a 2D section of the relevant supercrystal, \mathcal{S} . On the other hand, the projection, S_p , of the octagonal SQC onto the same line is identical to $S_s/\sqrt{2}$, which is a section of a 2D limit-periodic structure given as a 2D projection of \mathcal{S} . Conversely, a 2D SQC can be constructed by the grid method from a 1D LQPS. It is readily shown that the relevant hyperatoms are polygons.

The present theory is readily extended to the case of the icosahedral point group in 3D. Only the type I SSMs concern this case, and the index of an icosahedral SSM, $\nu\Lambda$, is given by $m = [N(\nu)]^3$.

There exists a bijection between the infinite set, \mathcal{B} , and the set of all the super-Bravais-lattices on a single-Bravais-lattice, $\hat{\Lambda}$, which is the host lattice of QCs. This means that the world of SQCs is far richer than that of QCs, which is contrary to a previous conception. SQCs together with QCs form an important class of aperiodic-ordered structures with non-crystallographic point symmetries.

Acknowledgment

The authors are grateful to O Terasaki.

References

- [1] Shechtman D, Blech I, Gratius D and Cahn J W 1984 *Phys. Rev. Lett.* **53** 1951
- [2] Penrose R 1974 *Bull. Inst. Math. Appl.* **10** 266
- [3] Levine D and Steinhardt P J 1984 *Phys. Rev. Lett.* **53** 2477
- [4] Katz A and Duneau M 1986 *J. Phys.* **47** 181
- [5] Bak P 1986 *Scr. Metall.* **20** 1199
- [6] Gaehler F and Klitzing R 1997 *The Mathematics of Long-Range Aperiodic Order (NATO ASI Series C vol 489)* ed R V Moody (Dordrecht: Kluwer) pp 141–74
- [7] Luck J M, Godrèche C, Janner A and Janssen T 1993 *J. Phys. A: Math. Gen.* **26** 1951
- [8] Kolář M 1993 *Phys. Rev. B* **47** 5489
- [9] Baake M, Moody R V and Schlottmann M 1998 *J. Phys. A: Math. Gen.* **31** 5755
- [10] Yamamoto A 1996 *Acta Cryst. A* **52** 509
- [11] Niizeki K 1989 *J. Phys. A: Math. Gen.* **22** 193
- [12] Niizeki K 1989 *J. Phys. A: Math. Gen.* **22** 205
- [13] Baake M, Schlottmann M and Jarvis P D 1991 *J. Phys. A: Math. Gen.* **24** 4637
- [14] Niizeki K 1989 *J. Phys. A: Math. Gen.* **22** 1859
- [15] Watanabe Y, Ito M and Soma T 1987 *Acta Cryst. A* **43** 133